Calculating two-point resistances in distance-regular resistor networks

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Abstract

An algorithm for the calculation of the resistance between two arbitrary nodes in an arbitrary distance-regular resistor network is provided, where the calculation is based on stratification introduced in Jafarizadeh and Salimi (2006 *J. Phys. A: Math. Gen.* **39** 1–29) and the Stieltjes transform of the spectral distribution (Stieltjes function) associated with the network. It is shown that the resistances between a node α and all nodes β belonging to the same stratum with respect to the α ($R_{\alpha\beta^{(i)}}$, β belonging to the ith stratum with respect to the α) are the same. Also, the analytical formulae for two-point resistances $R_{\alpha\beta^{(i)}}$, $i = 1, 2, 3$, are given in terms of the size of the network and corresponding intersection numbers. In particular, the two-point resistances in a strongly regular network are given in terms of its parameters (v, κ, λ, μ). Moreover, the lower and upper bounds for two-point resistances in strongly regular networks are discussed.

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1. Introduction

A classic problem in electric circuit theory studied by numerous authors over many years is the computation of the resistance between two nodes in a resistor network (see, e.g., [2]). Besides being a central problem in electric circuit theory, the computation of resistances is also relevant to a wide range of problems ranging from random walks (see [3]), the theory of harmonic functions [4], to lattice Greens functions [5–9]. The connection with these problems originates from the fact that electrical potentials on a grid are governed by the same difference equations as those occurring in the other problems. For this reason, the resistance problem is often studied from the point of view of solving the difference equations, which is most conveniently carried out for infinite networks. In the case of Greens function approach, for example, past efforts [2, 10] have been focused mainly on infinite lattices. Little attention has been paid to

finite networks, even though the latter are those occurring in real life. In this paper, we take up this problem and present a general formulation for computing two-point resistances in finite networks. Particularly, we show that the known results for infinite networks are recovered by taking the infinite-size limit.

The study of electric networks was formulated by Kirchhoff [11] more than 150 years ago as an instance of a linear analysis. Our starting point is along the same line by considering the Laplacian matrix associated with a network. The Laplacian is a matrix whose off-diagonal entries are the conductances connecting pairs of nodes. Just as in graph theory where everything about a graph is described by its adjacency matrix (whose element is 1 if two vertices are connected and 0 otherwise), everything about an electric network is described by its Laplacian. The author of [12] has derived an expression for the two-point resistance between two arbitrary nodes α and β of a regular network in terms of the matrix entries $L_{\alpha\alpha}^{-1}$, $L_{\beta\beta}^{-1}$ and $L_{\alpha\beta}^{-1}$, where L^{-1} is the pseudo-inverse of the Laplacian matrix. Here in this work, based on stratification introduced in [1] and spectral analysis method, we introduce a procedure for calculating two-point resistances in distance-regular resistor networks in terms of the Stieltjes function $G_\mu(x)$ associated with the adjacency matrix of the network and its derivatives. Although we discuss the case of distance-regular networks, the method can also be used for any arbitrary regular network. It should be noted that, in this way, the two-point resistances are calculated straightforwardly without any need to know the spectrum of the network. Also, it is shown that the resistances between a node α and all nodes β belonging to the same stratum with respect to the α ($R_{\alpha\beta^{(i)}}$, β belonging to the ith stratum with respect to the α) are the same. We give the analytical formulae for two-point resistances $R_{\alpha\beta^{(i)}}$, $i = 1, 2, 3$, in terms of the network's characteristics such as the size of the network and its intersection array. In particular, the two-point resistances in a strongly regular network are given in terms of the network's parameters $(v, \kappa, \lambda, \mu)$. Moreover, we discuss the lower and upper bounds for two-point resistances in strongly regular networks. From the fact that, the two-point resistances on a network depend on the corresponding Stieltjes function $G_\mu(x)$ and that $G_\mu(x)$ is written as a continued fraction, the two-point resistances on an infinite-size network can be approximated with those of the corresponding finite-size networks.

The organization of the paper is as follows. In section 2, we give some preliminaries such as association schemes, distance-regular networks, stratification of these networks and Stieltjes function associated with the network. In section 3, two-point resistances in distance-regular networks are given in terms of the Stieltjes function and its derivatives. Also, the resistances $R_{\alpha\beta^{(i)}}$, $i = 1, 2, 3$, are given in terms of the network's intersection array. In particular, two-point resistances in a strongly regular network are given in terms of the network's parameters; also lower and upper bounds for the two-point resistances in these networks are discussed. Section 4 is devoted to calculating two-point resistances $R_{\alpha\beta^{(i)}}$, $i = 1, 2, 3$, in some important examples of distance-regular networks, such as complete network, strongly regular networks (distance-regular networks with diameter 2), e.g. Petersen and normal subgroup scheme networks [1], d-cube (d dimensional hypercube) and Johnson networks. The paper is ended with a brief conclusion and an appendix containing a table for two-point resistances $R_{\alpha\beta^{(i)}}$, $i = 1, 2, 3$, of some important distance-regular resistor networks with the size less than 70.

2. Preliminaries

In this section, we give some preliminaries such as definitions related to association schemes, corresponding stratification, distance-regular networks and Stieltjes function associated with a distance-regular network.
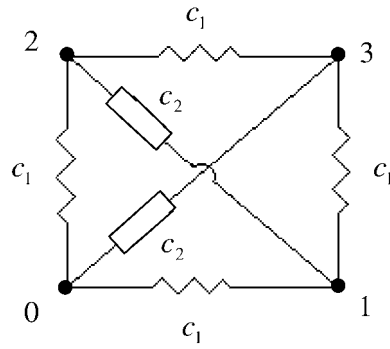
Figure 1. Shows a resistor network with two kinds of conductances corresponding to two kinds of relations R_1 and R_2.

2.1. Association schemes

First we recall the definition of association schemes. The reader is referred to [14] for further information on association schemes.

Definition 2.1 (symmetric association schemes). *Let V be a set of vertices, and let R_i ($i = 0, 1, \ldots, d$) be nonempty relations on V (i.e., subset of $V \times V$). Let the following conditions (1), (2), (3) and (4) be satisfied. Then, the relations $\{R_i\}_{0 \leqslant i \leqslant d}$ on $V \times V$ satisfying the following conditions,*

(1) $\{R_i\}_{0 \leqslant i \leqslant d}$ is a partition of $V \times V$
(2) $R_0 = \{(\alpha, \alpha) : \alpha \in V\}$
(3) $R_i = R_i^t$ for $0 \leqslant i \leqslant d$, where $R_i^t = \{(\beta, \alpha) : (\alpha, \beta) \in R_i\}$
(4) For $(\alpha, \beta) \in R_k$, the number $p_{ij}^k = |\{\gamma \in V : (\alpha, \gamma) \in R_i \text{ and } (\gamma, \beta) \in R_j\}|$ does not depend on (α, β) but only on i, j and k,

define a symmetric association scheme of class d on V which is denoted by $Y = (V, \{R_i\}_{0 \leqslant i \leqslant d})$. Furthermore, if we have $p_{ij}^k = p_{ji}^k$ for all $i, j, k = 0, 2, \ldots, d$, then Y is called commutative.

The number v of the vertices, $|V|$, is called the order of the association scheme and R_i is called ith relation. For example, in the resistor networks the relations R_i, $i = 0, 1, \ldots, d$, can be interpreted as $d + 1$ different kinds of conductances, i.e., two nodes α, β have the ith relation with each other if and only if the conductance between them be c_i (see figure 1). In this paper, we will deal with the special case where the conductance between two nodes α, β is $c \equiv c_1$ if the nodes be adjacent, i.e., $(\alpha, \beta) \in R_1$ and the other conductances corresponding to the other relations will be taken to zero.

Also note that, the intersection number p_{ij}^k can be interpreted as the number of vertices which have relation i and j with vertices α and β, respectively, provided that $(\alpha, \beta) \in R_k$, and it is the same for all elements of relation R_k. For all integers i ($0 \leqslant i \leqslant d$), set $\kappa_i = p_{ii}^0$ and note that $\kappa_i \neq 0$, since R_i is non-empty. We refer to κ_i as the ith valency of Y.

For examples of association schemes, consider a cube known as Hamming scheme $H(3, 2)$, in which V (the vertex set) is the set of 3-tuples with entries in $F_2 = \{0, 1\}$. Two vertices are connected if and only if they differ by exactly one entry (see figure 2). The distance between vertices, i.e. the length of the shortest edge path connecting them, will then indicate which relation they are contained in. For example, if $x = (0, 0, 1)$, $y = (0, 1, 1)$ and $z = (1, 0, 1)$, then $(x, y) \in R_1$, $(x, z) \in R_1$ and $(y, z) \in R_2$. As an another example, consider

0-th stratum   1-th stratum   2-th stratum   3-th stratum
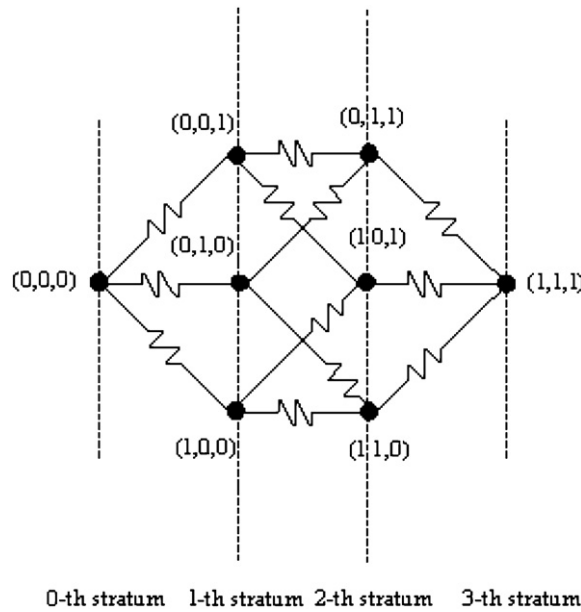
Figure 2. (*a*) Shows the cube or Hamming scheme $H(3, 2)$ with a vertex set $V = \{(ijk) : i, j, k = 0, 1\}$ and relations $R_0 = \{((ijk), (ijk)) : (ijk) \in V\}$, $R_1 = \{((ijk), (i'jk)), ((ijk), (ij'k)), ((ijk), (ijk')) : i \neq i', j \neq j', k \neq k'\}$, $R_2 = \{((ijk), (i'j'k)), ((ijk), (i'jk')), ((ijk), (ij'k')) : i \neq i', j \neq j', k \neq k'\}$ and $R_3 = \{((ijk), (i'j'k')) : i \neq i', j \neq j', k \neq k'\}$ respectively. Its non-vanishing intersection numbers are $p_{11}^0 = 3$, $p_{11}^2 = 2$, $p_{12}^1 = p_{21}^1 = 2$, $p_{12}^3 = p_{21}^3 = 3$, $p_{13}^2 = p_{31}^2 = 1$, $p_{22}^0 = 3$, $p_{22}^0 = 3$, $p_{23}^1 = p_{32}^1 = 1$, $p_{33}^0 = 1$. (*b*) The vertical dashed lines denote the four strata of the cube.
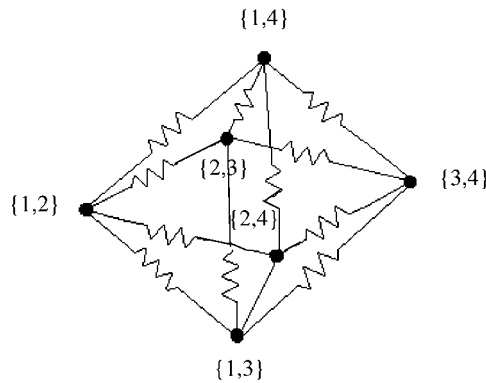
Figure 3. Shows the octahedron or Johnson scheme $J(4, 2)$.

the octahedron (a special case of a square dipyramid with equal edge lengths) which is the same as the Johnson scheme $J(4, 2)$ in which the vertex set V contains all two-element subsets of the set $\{1, 2, 3, 4\}$ and two vertices are adjacent if and only if they intersect in exactly one element. Two vertices are then at distance i, $i = 0, 1, 2$, if and only if they have exactly $2 - i$ elements in common (see figure 3).
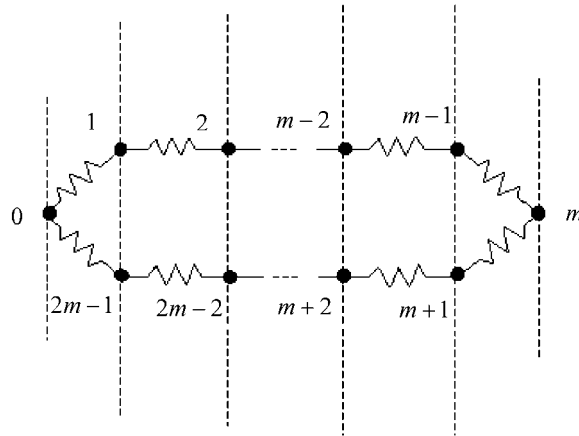
Figure 4. Denotes the cycle resistor network C_{2m}, where the $m + 1$ vertical dashed lines show the $m + 1$ strata of the network.

Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a commutative symmetric association scheme of class d, then the matrices $A_0, A_1, \ldots, A_d$ defined by

$$(A_i)_{\alpha, \beta} = \begin{cases} 1 & \text{if} \quad (\alpha, \beta) \in R_i \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

are adjacency matrices of Y such that

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k. \tag{2.2}$$

From (2.2), it is seen that the adjacency matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative algebra A known as the Bose–Mesner algebra of Y. For example, consider the cycle graph with v vertices denoted by C_v (see figure 4 for even $v = 2m$). From figure 4, it can be easily seen that, for even number of vertices $v = 2m$, the adjacency matrices are given by

$$A_i = S^i + S^{-i}, \qquad i = 1, 2, \ldots, m - 1, \qquad A_m = S^m, \tag{2.3}$$

where S is the $v \times v$ circulant matrix with period v ($S^v = I_v$) defined as follows:

$$S = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{2.4}$$

For odd number of vertices $v = 2m + 1$, we have

$$A_i = S^i + S^{-i}, \qquad i = 1, 2, \ldots, m - 1, m. \tag{2.5}$$

One can easily check that the adjacency matrices in (2.3) together with $A_0 = I_{2m}$ (and also the adjacency matrices in (2.5) together with $A_0 = I_{2m+1}$) form a commutative algebra.

Finally, the underlying graph of an association scheme $\Gamma = (V, R_1)$ is an undirected connected graph, where the sets V and R_1 consist of their vertices and edges, respectively. Obviously replacing R_1 with one of the other relations $R_i, i \neq 0, 1$, will also give us an underlying graph $\Gamma = (V, R_i)$ (not necessarily a connected graph) with the same set of
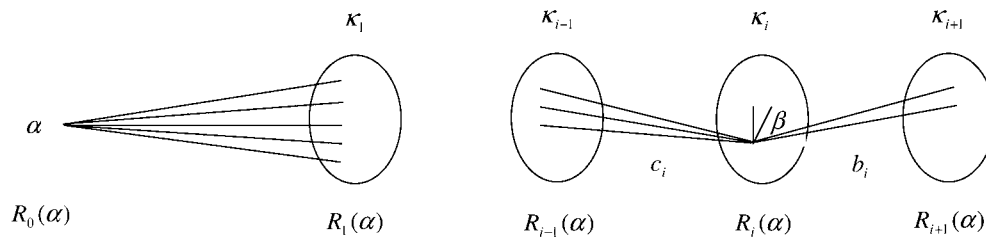
Figure 5. Shows edges through α and β in a distance-regular graph.

vertices but a new set of edges R_i. For example, the cube is the underlying graph of the Hamming scheme $H(3, 2)$ with the adjacency matrices (other than $A_0 = I \otimes I \otimes I$) as follows:

$$A_1 = \sigma_x \otimes I \otimes I + I \otimes \sigma_x \otimes I + I \otimes I \otimes \sigma_x,$$

$$A_2 = \sigma_x \otimes \sigma_x \otimes I + I \otimes \sigma_x \otimes \sigma_x + \sigma_x \otimes I \otimes \sigma_x, \tag{2.6}$$

$$A_3 = \sigma_x \otimes \sigma_x \otimes \sigma_x,$$

where I is the 2×2 unit matrix and σ_x is the Pauli matrix.

2.2. Stratifications

For a given vertex $\alpha \in V$, let $R_i(\alpha) := \{\beta \in V : (\alpha, \beta) \in R_i\}$ denote the set of all vertices having the relation R_i with α. Then, the vertex set V can be written as a disjoint union of $R_i(\alpha)$ for $i = 0, 1, 2, \ldots, d$, i.e.,

$$V = \bigcup_{i=0}^{d} R_i(\alpha). \tag{2.7}$$

We fix a point $o \in V$ as an origin of the underlying graph of an association scheme, called the reference vertex. Then, relation (2.7) stratifies the underlying graph into a disjoint union of associate classes $R_i(o)$ (called the ith stratum with respect to o). Let $l^2(V)$ denote the Hilbert space of C-valued square-summable functions on V. With each associate class $R_i(o)$, we associate a unit vector in $l^2(V)$ defined by

$$|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} \sum_{\alpha \in R_i(o)} |\alpha\rangle, \tag{2.8}$$

where $|\alpha\rangle$ denotes the eigenket of the αth vertex at the associate class $R_i(o)$ and $\kappa_i = |R_i(o)|$ is called the ith valency of the graph $\left(\kappa_i := p_{ii}^0 = |\{\gamma : (o, \gamma) \in R_i\}| = |R_i(o)|\right)$. The closed subspace of $l^2(V)$ spanned by $\{|\phi_i\rangle\}$ is denoted by $\Lambda(G)$. Since $\{|\phi_i\rangle\}$ becomes a complete orthonormal basis of $\Lambda(G)$, we often write

$$\Lambda(G) = \sum_{i} \oplus \mathbf{C} |\phi_i\rangle. \tag{2.9}$$

Let A_i be the adjacency matrix of the graph $\Gamma = (V, R)$. Then, from the definition of the ith adjacency matrix A_i, for the reference state $|\phi_0\rangle$ ($|\phi_0\rangle = |o\rangle$, with $o \in V$ as reference vertex), we have

$$A_i |\phi_0\rangle = \sum_{\beta \in R_i(o)} |\beta\rangle. \tag{2.10}$$

Then by using (2.8) and (2.10), we have

$$A_i|\phi_0\rangle = \sqrt{\kappa_i}|\phi_i\rangle. \tag{2.11}$$

For the sake of clarity, consider the cube and the cycle graph C_{2m} with the adjacency matrices as in (2.6) and (2.3), respectively. Then, by using (2.11) and choosing $|000\rangle$ as the origin (reference vertex), the cube is stratified to four strata (see figure 2) such that the unit vectors are given by

$$|\phi_0\rangle = |000\rangle, \qquad |\phi_1\rangle = \tfrac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),$$
$$|\phi_2\rangle = \tfrac{1}{\sqrt{3}}(|011\rangle + |110\rangle + |101\rangle), \qquad |\phi_3\rangle = |111\rangle, \tag{2.12}$$

where by choosing $|0\rangle$ as the origin, the cycle graph C_{2m} is stratified to $m + 1$ strata (see figure 4) such that the unit vectors are given by

$$|\phi_0\rangle = |0\rangle, \qquad |\phi_i\rangle = \tfrac{1}{\sqrt{2}}(|i\rangle + |2m - i\rangle), \qquad i = 1, 2, \ldots, m - 1, \qquad |\phi_m\rangle = |m\rangle. \tag{2.13}$$

2.3. Distance-regular graphs

In this section, we consider some set of important graphs called distance-regular graphs. First, we recall the definition of so-called P-polynomial association schemes (which are closely related to the distance-regular graphs) as follows.

Definition 2.2 (*P*-polynomial property). *The symmetric association scheme $Y = (X, \{R_i\}_{0\leqslant i\leqslant d})$ is said to be P-polynomial (with respect to the ordering $R_0, \ldots, R_d$ of the associate classes) whenever for all $i = 0, 1, \ldots, d$, there exist $d_i, e_i, f_i; d_i \neq 0 \neq f_i$ with*

$$A_1 A_i = d_i A_{i-1} + e_i A_i + f_i A_{i+1}. \tag{2.14}$$

Condition (2.14) is similar to the well-known three-term recursion relations appearing in the theory of orthogonal polynomials, where A_1 is in correspondence with x (see equation (2.36) in subsection 2.4). Using the recursion relations (2.14), one can show that A_i is a polynomial in A_1 of degree i, i.e., we have $A_i = P_i(A_1), i = 1, 2, \ldots, d$. In particular, $A \equiv A_1$ multiplicatively generates the Bose–Mesner algebra (for more details see [15]).

An undirected connected graph $\Gamma = (V, R_1)$ is called the distance-regular graph if it is the underlying graph of a P-polynomial association scheme, where the relations are based on a distance function defined as follows. Let the distance between $\alpha, \beta \in V$ denoted by $\partial(\alpha, \beta)$ is the length of the shortest walk connecting α and β (recall that a finite sequence $\alpha_0, \alpha_1, \ldots, \alpha_n \in V$ is called a walk of length n if $\alpha_{k-1} \sim \alpha_k$ for all $k = 1, 2, \ldots, n$, where $\alpha_{k-1} \sim \alpha_k$ means that α_{k-1} is adjacent with α_k) then the relations R_i in distance-regular graphs are defined as $(\alpha, \beta) \in R_i$ if and only if $\partial(\alpha, \beta) = i, i = 0, 1, \ldots, d$, where $d := \max\{\partial(\alpha, \beta) : \alpha, \beta \in V\}$ is called the diameter of the graph. Since $\partial(\alpha, \beta)$ gives the distance between vertices α and β, ∂ is called the distance function. Clearly, we have $\partial(\alpha, \alpha) = 0$ for all $\alpha \in V$ and $\partial(\alpha, \beta) = 1$ if and only if $\alpha \sim \beta$. Therefore, distance-regular graphs become metric spaces with the distance function ∂.

One should note that, condition (2.14) implies that for distance-regular graphs, we have the following relation:

$$R_1(\beta) \subseteq R_{i-1}(\alpha) \cup R_i(\alpha) \cup R_{i+1}(\alpha), \qquad \forall \beta \in R_i(\alpha). \tag{2.15}$$

We also note that, in distance-regular graphs, the stratification is reference vertex independent, namely one can choose every vertex as a reference one, while the stratification of more general graphs may be reference dependent.

Relation (2.14) implies that in a distance-regular graph, $p_{j1}^i = 0$ (for $i \neq 0$, j dose not belong to $\{i - 1, i, i + 1\}$), i.e., the nonzero intersection numbers of the graph are given by

$$a_i = p_{i1}^i, \qquad b_i = p_{i+1,1}^i, \qquad c_i = p_{i-1,1}^i, \tag{2.16}$$

respectively (see figure 5). The intersection numbers (2.16) and the valencies κ_i satisfy the following obvious conditions:

$$\begin{aligned} a_i + b_i + c_i = \kappa, \qquad & \kappa_{i-1} b_{i-1} = \kappa_i c_i, \qquad && i = 1, \ldots, d, \\ \kappa_0 = c_1 = 1, \qquad & b_0 = \kappa_1 = \kappa, \qquad && (c_0 = b_d = 0). \end{aligned} \tag{2.17}$$

Thus all parameters of the graph can be obtained from the intersection array $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$.

By using equations (2.2) and (2.17), for adjacency matrices of the distance-regular graph Γ, we obtain

$$\begin{aligned} A_1 A_i &= b_{i-1} A_{i-1} + (\kappa - b_i - c_i) A_i + c_{i+1} A_{i+1}, \qquad i = 1, 2, \ldots, d - 1, \\ A_1 A_d &= b_{d-1} A_{d-1} + (\kappa - c_d) A_d. \end{aligned} \tag{2.18}$$

The recursion relations (2.18) imply that

$$A_i = P_i(A), \qquad i = 0, 1, \ldots, d. \tag{2.19}$$

By acting two sides of (2.18) on $|\phi_0\rangle$ and using (2.11), we obtain

$$\sqrt{\kappa_i} A |\phi_i\rangle = \sqrt{\kappa_{i-1}} b_{i-1} |\phi_{i-1}\rangle + \sqrt{\kappa_i} \alpha_i |\phi_i\rangle + \sqrt{\kappa_{i+1}} c_{i+1} |\phi_{i+1}\rangle, \qquad i = 0, 1, \ldots, d. \tag{2.20}$$

Then, by dividing the sides of (2.20) by $\sqrt{\kappa_i}$ and using (2.17), one can easily obtain the following three-term recursion relations for the unit vectors $|\phi_i\rangle$, $i = 0, 1, \ldots, d$:

$$A |\phi_i\rangle = \beta_{i+1} |\phi_{i+1}\rangle + \alpha_i |\phi_i\rangle + \beta_i |\phi_{i-1}\rangle, \tag{2.21}$$

where, the coefficients α_i and β_i are defined as

$$\alpha_0 = 0, \qquad \alpha_k = \kappa - b_k - c_k, \qquad \omega_k \equiv \beta_k^2 = b_{k-1} c_k, \qquad k = 1, \ldots, d. \tag{2.22}$$

That is, in the basis of the unit vectors $\{|\phi_i\rangle, i = 0, 1, \ldots, d\}$, the adjacency matrix A is projected to the following symmetric tridiagonal form:

$$A = \begin{pmatrix} \alpha_0 & \beta_1 & 0 & \ldots & \ldots & 0 \\ \beta_1 & \alpha_1 & \beta_2 & 0 & \ldots & 0 \\ 0 & \beta_2 & \alpha_3 & \beta_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \beta_{d-1} & \alpha_{d-1} & \beta_d \\ 0 & \ldots & 0 & 0 & \beta_d & \alpha_d \end{pmatrix}. \tag{2.23}$$

In [26], it has been shown that the coefficients α_i and β_i can also be obtained easily by using the Lanczos iteration algorithm. We will refer to the parameters α_i and ω_i defined in (2.22) as QD (quantum decomposition) parameters.

A well-known example of distance-regular graphs is the cycle graph C_v. By using (2.3) and (2.5), one can obtain the following recursion relations for C_{2m} and C_{2m+1}:

$$A_1 A_i = A_{i-1} + A_{i+1}, \qquad i = 0, 1, \ldots, m - 1; \qquad A_1 A_m = A_{m-1} \tag{2.24}$$

and

$$A_1 A_i = A_{i-1} + A_{i+1}, \qquad i = 0, 1, \ldots, m - 1; \qquad A_1 A_m = A_{m-1} + A_m, \tag{2.25}$$

respectively (the graph C_v for $v = 2m$ or $v = 2m + 1$ consists of $m + 1$ strata). By comparing (2.24) and (2.25) with the three-term recursion relations (2.18), we obtain the intersection arrays for even and odd number of vertices as

$$\{b_0, \ldots, b_{m-1}; c_1, \ldots, c_m\} = \{2, 1, \ldots, 1, 1; 1, \ldots, 1, 2\} \qquad (2.26)$$

and

$$\{b_0, \ldots, b_{m-1}; c_1, \ldots, c_m\} = \{2, 1, \ldots, 1; 1, \ldots, 1, 1\}, \qquad (2.27)$$

respectively. Then, by using (2.22), for even $v = 2m$ the QD parameters are given by

$$\alpha_i = 0, \qquad i = 0, 1, \ldots, m; \qquad \omega_1 = \omega_m = 2, \qquad \omega_i = 1, \qquad i = 2, \ldots, m - 1, \qquad (2.28)$$

where, for odd $v = 2m + 1$, we obtain

$$\alpha_i = 0, \qquad i = 0, 1, \ldots, m - 1, \qquad \alpha_m = 1; \qquad \omega_1 = 2, \quad \omega_i = 1, \quad i = 2, \ldots, m. \qquad (2.29)$$

2.4. Stieltjes function associated with the network

In this subsection, we recall the definition of the Stieltjes function associated with a distance-regular network which is related to the spectral distribution corresponding to the network. To do so, first we recall some facts about the spectral distribution associated with the adjacency matrix of the network. In fact, the spectral analysis of operators is an important issue in quantum mechanics, operator theory and mathematical physics [16, 17]. Since the advent of random matrix theory (RMT), there has been considerable interest in the statistical analysis of spectra [18–20]. RMT can be viewed as a generalization of the classical probability calculus, where the concept of probability density distribution for a one-dimensional random variable is generalized onto an averaged spectral distribution of the ensemble of large, non-commuting random matrices. Such a structure exhibits several phenomena known in classical probability theory, including central limit theorems [21]. Also, the two-point resistance has a probabilistic interpretation based on classical random walker walking on the network. Indeed, the connection between random walks and electric networks has been recognized for some time (see e.g. [22–24]), where one can establish a connection between the electrical concepts of current and voltage and corresponding descriptive quantities of random walks regarded as finite state Markov chains (for more details see [3]). Also, by adapting the random-walk dynamics and mean-field theory it has been studied that [25], how the growth of a conducting network, such as electrical or electronic circuits, interferes with the current flow through the underlying evolving graphs. In this paper, it is shown that there is also a connection between the mathematical techniques introduced in previous subsections and this subsection such as Hilbert space of the stratification and spectral techniques (which have been employed in [1, 26] for investigating continuous time quantum walk on graphs), and electrical concept of resistance between two arbitrary nodes of regular networks and so the same techniques can be used for calculating the resistance. Note that, although we take the spectral approach to define the Stieltjes function in terms of orthogonal polynomials (which are orthogonal with respect to the spectral distribution μ associated with the network) with three-term recursion relations, in practice as will be seen in section 3, we will calculate two-point resistances without any need to evaluate the spectral distribution μ.

It is well known that, for any pair $(A, |\phi_0\rangle)$ of a matrix A and a vector $|\phi_0\rangle$, it can be assigned a measure μ as follows:

$$\mu(x) = \langle\phi_0|E(x)|\phi_0\rangle, \qquad (2.30)$$

where $E(x) = \sum_i |u_i\rangle\langle u_i|$ is the operator of projection onto the eigenspace of A corresponding to eigenvalue x, i.e.,

$$A = \int x E(x)\, \mathrm{d}x. \tag{2.31}$$

It is easy to see that, for any polynomial $P(A)$ we have

$$P(A) = \int P(x) E(x)\, \mathrm{d}x, \tag{2.32}$$

where for discrete spectrum the above integrals are replaced by summation. Therefore, using relations (2.30) and (2.32), the expectation value of powers of adjacency matrix A over starting site $|\phi_0\rangle$ can be written as

$$\langle\phi_0|A^m|\phi_0\rangle = \int_R x^m \mu(\mathrm{d}x), \qquad m = 0, 1, 2, \ldots. \tag{2.33}$$

The existence of a spectral distribution satisfying (2.33) is a consequence of Hamburgers theorem, see e.g., [27, theorem 1.2].

Obviously relation (2.33) implies an isomorphism from the Hilbert space of the stratification onto the closed linear span of the orthogonal polynomials with respect to the measure μ. More clearly, the orthonormality of the unit vectors $|\phi_i\rangle$ implies that

$$\delta_{ij} = \langle\phi_i|\phi_j\rangle = \frac{1}{\sqrt{\kappa_i \kappa_j}}\langle\phi_0|A_i A_j|\phi_0\rangle = \int_R P_i'(x) P_j'(x)\mu(\mathrm{d}x), \tag{2.34}$$

where we have used equations (2.11) and (2.19) to write

$$|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} A_i|\phi_0\rangle = \frac{1}{\sqrt{\kappa_i}} P_i(A)|\phi_0\rangle \equiv P_i'(A)|\phi_0\rangle, \tag{2.35}$$

with $P_i'(A) := \frac{1}{\sqrt{\kappa_i}} P_i(A)$. Now, by substituting (2.35) in (2.21), we get three-term recursion relations between polynomials $P_j'(A)$, which lead to the following three term recursion relations between polynomials $P_j'(x)$:

$$x P_k'(x) = \beta_{k+1} P_{k+1}'(x) + \alpha_k P_k'(x) + \beta_k P_{k-1}'(x) \tag{2.36}$$

for $k = 0, \ldots, d-1$, with $P_0'(x) = 1$. Multiplying two sides of (2.36) by $\beta_1 \ldots \beta_k$ we obtain

$$\beta_1 \ldots \beta_k x P_k'(x) = \beta_1 \ldots \beta_{k+1} P_{k+1}'(x) + \alpha_k \beta_1 \ldots \beta_k P_k'(x) + \beta_k^2.\beta_1 \ldots \beta_{k-1} P_{k-1}'(x). \tag{2.37}$$

By rescaling P_k' as $Q_k = \beta_1 \ldots \beta_k P_k'$, the spectral distribution μ under question is characterized by the property of orthonormal polynomials $\{Q_k\}$ defined recurrently by

$$\begin{aligned} Q_0(x) &= 1, \qquad Q_1(x) = x, \\ x Q_k(x) &= Q_{k+1}(x) + \alpha_k Q_k(x) + \beta_k^2 Q_{k-1}(x), \qquad k \geqslant 1. \end{aligned} \tag{2.38}$$

If such a spectral distribution is unique, the spectral distribution μ is determined by the identity

$$G_\mu(x) = \int_R \frac{\mu(\mathrm{d}y)}{x - y} = \cfrac{1}{x - \alpha_0 - \cfrac{\beta_1^2}{x - \alpha_1 - \cfrac{\beta_2^2}{x - \alpha_2 - \cfrac{\beta_3^2}{x - \alpha_3 - \cdots}}}} = \frac{Q_{d-1}^{(1)}(x)}{Q_d(x)} = \sum_{l=0}^{d-1} \frac{A_l}{x - x_l}, \tag{2.39}$$

where x_l are the roots of polynomial $Q_d(x)$. $G_\mu(x)$ is called the Stieltjes/Hilbert transform of spectral distribution μ or the Stieltjes function and polynomials $\{Q_k^{(1)}\}$ are defined recurrently as

$$\begin{aligned} Q_0^{(1)}(x) &= 1, \qquad Q_1^{(1)}(x) = x - \alpha_1, \\ x Q_k^{(1)}(x) &= Q_{k+1}^{(1)}(x) + \alpha_{k+1} Q_k^{(1)}(x) + \beta_{k+1}^2 Q_{k-1}^{(1)}(x), \qquad k \geqslant 1, \end{aligned} \tag{2.40}$$

respectively. The coefficients A_l appearing in (2.39) are calculated as

$$A_l = \lim_{x \to x_l} (x - x_l) G_\mu(x) \tag{2.41}$$

(for more details see [27–30]).

Note that the third equality in (2.39) is a well-known result in the theory of continued fractions (see for example [13]). Historically, the orthogonal polynomials originated in the theory of continued fractions. This relationship is of great importance and is one of the possible starting points of the treatment of orthogonal polynomials.

3. Two-point resistances in regular resistor networks

A classic problem in electric circuit theory studied by numerous authors over many years is the computation of the resistance between two nodes in a resistor network (see, e.g., [2]). The results obtained in this section show that there is a close connection between the techniques introduced in section 2 such as Hilbert space of the stratification and the Stieltjes function and electrical concept of resistance between two arbitrary nodes of regular networks, and these techniques can be employed for calculating two-point resistances.

For a given regular graph Γ with v vertices and adjacency matrix A, let $r_{ij} = r_{ji}$ be the resistance of the resistor connecting vertices i and j. Hence, the conductance is $c_{ij} = r_{ij}^{-1} = c_{ji}$ so that $c_{ij} = 0$ if there is no resistor connecting i and j. Denote the electric potential at the ith vertex by V_i and the net current flowing into the network at the ith vertex by I_i (which is zero if the ith vertex is not connected to the external world). Since there exist no sinks or sources of current including the external world, we have the constraint $\sum_{i=1}^{v} I_i = 0$. The Kirchhoff law states

$$\sum_{j=1, j \neq i}^{v} c_{ij}(V_i - V_j) = I_i, \qquad i = 1, 2, \ldots, v. \tag{3.1}$$

Explicitly, equation (3.1) reads

$$L\vec{V} = \vec{I}, \tag{3.2}$$

where $\vec{V}$ and $\vec{I}$ are v-vectors whose components are V_i and I_i, respectively, and

$$L = \sum_i c_i |i\rangle\langle i| - \sum_{i,j} c_{ij} |i\rangle\langle j| \tag{3.3}$$

is the Laplacian of the graph Γ with

$$c_i \equiv \sum_{j=1, j \neq i}^{v} c_{ij}, \tag{3.4}$$

for each vertex α. Hereafter, we will assume that all nonzero resistances are equal to r, then the off-diagonal elements of $-L$ are precisely those of $\frac{1}{r} A$, i.e.,

$$L = \frac{1}{r}(\kappa I - A), \tag{3.5}$$

with $\kappa = \deg(\alpha)$, for each vertex α. It should be noted that L has the eigenvector $(1, 1, \ldots, 1)^t$ with the eigenvalue 0. Therefore, L is not invertible and so we define the pseudo-inverse of L as

$$L^{-1} = \sum_{i, \lambda_i \neq 0} \lambda_i^{-1} E_i, \tag{3.6}$$

where E_i is the operator of projection onto the eigenspace of L^{-1} corresponding to the eigenvalue λ_i. Following the result of [12] and that L^{-1} is a real matrix, the resistance between vertices α and β is given by

$$R_{\alpha\beta} = \langle\alpha|L^{-1}|\alpha\rangle - 2\langle\alpha|L^{-1}|\beta\rangle + \langle\beta|L^{-1}|\beta\rangle. \tag{3.7}$$

In this paper, we will consider distance-regular graphs as resistor networks. Then, the diagonal entries of L^{-1} are independent of the vertex, i.e., $L_{\alpha\alpha}^{-1} = L_{\beta\beta}^{-1}$ for all $\alpha, \beta \in V$. Therefore, from relation (3.7), we can obtain the two-point resistance between two arbitrary nodes α and β as follows:

$$R_{\alpha\beta} = 2\big(L_{\alpha\alpha}^{-1} - L_{\alpha\beta}^{-1}\big). \tag{3.8}$$

Therefore, for evaluating two-point resistance $R_{\alpha\beta}$, we need to calculate the matrix entries $L_{\alpha\alpha}^{-1}$ and $L_{\alpha\beta}^{-1}$. To this end, as equation (3.6) shows, one needs to know the spectrum of the pseudo-inverse L^{-1} which is a task with high complexity for networks with a large size, even with computer. In the following, we give a method such that the two-point resistances are calculated without any knowledge about the spectrum of the pseudo-inverse of Laplacian of the network. In this method, we need only the Stieltjes function of the network which is calculated easily via (2.39) by using the QD parameters of the network and recursion relations (2.38) and (2.40). In fact, by using this method, we give explicit analytical formulae for two-point resistances $R_{\alpha\beta^{(i)}}$, $i = 1, 2, 3$; ($R_{\alpha\beta^{(k)}}$ denotes the two-point resistance between α as reference node and each $\beta \in R_k(\alpha)$) on any distance-regular network in terms only of the intersection array of the network without any need to evaluate the spectrum of L^{-1}.

Let α and β be two arbitrary nodes of the network such that β belongs to the lth stratum with respect to α, i.e., $\beta \in R_l(\alpha)$ (we choose one of the nodes, here α, as reference node). Then, for calculating the matrix entries $L_{\alpha\alpha}^{-1}$ and $L_{\beta\alpha}^{-1}$ in (3.7), we use the Stieltjes function to obtain

$$L_{\alpha\alpha}^{-1} = r\langle\alpha|\frac{1}{\kappa I - A}|\alpha\rangle = r\int_{R-\{\kappa\}} \frac{d\mu(x)}{\kappa - x} = r\sum_{i,i\neq0}^{d-1} \frac{A_i}{\kappa - x_i} = r\lim_{y\to\kappa}\left(G_\mu(y) - \frac{A_0}{y - \kappa}\right) \tag{3.9}$$

and

$$\begin{aligned} L_{\beta\alpha}^{-1} &= r\langle\beta|\frac{1}{\kappa I - A}|\alpha\rangle = \frac{r}{\sqrt{\kappa_l}}\langle\phi_l|\frac{1}{\kappa I - A}|\alpha\rangle = \frac{r}{\sqrt{\kappa_l}}\langle\alpha|\frac{P_l'(A)}{\kappa I - A}|\alpha\rangle \\ &= \frac{r}{\sqrt{\kappa_l}}\int_{R-\{\kappa\}} \frac{d\mu(x)}{\kappa - x}P_l'(x) = \frac{r}{\sqrt{\kappa_l}}\sum_{i,i\neq0} \frac{A_i P_l'(x_i)}{\kappa - x_i}, \end{aligned} \tag{3.10}$$

where we have considered $x_0 = \kappa$ (κ is the eigenvalue corresponding to the idempotent E_0). Then, by using (3.8), the two-point resistance $R_{\alpha\beta^{(l)}}$ in the network is given by

$$R_{\alpha\beta^{(l)}} = \frac{2r}{\sqrt{\kappa_l}}\left\{\sqrt{\kappa_l}\lim_{y\to\kappa}\left(G_\mu(y) - \frac{A_0}{y - \kappa}\right) - \sum_{i,i\neq0} \frac{A_i P_l'(x_i)}{\kappa - x_i}\right\}. \tag{3.11}$$

For evaluating the term $\sum_{i,i\neq0} \frac{A_i P_l'(x_i)}{\kappa - x_i}$ in (3.11), we need to calculate

$$I_m := \sum_{i,i\neq0} \frac{A_i x_i^m}{\kappa - x_i}, \qquad \text{for} \quad m = 0, 1, \ldots, l. \tag{3.12}$$

To do so, we write the term (3.12) as

$$I_m = \sum_{i,i\neq 0} \frac{A_i x_i^m}{\kappa - x_i} = \sum_{i,i\neq 0} \frac{A_i\left((x_i - \kappa)^m - \sum_{l=1}^{m}(-1)^l C_l^m \kappa^l x_i^{m-l}\right)}{\kappa - x_i}$$

$$= -\sum_{i,i\neq 0} A_i (x_i - \kappa)^{m-1} - \sum_{l=1}^{m}(-1)^l C_l^m \kappa^l \sum_{i,i\neq 0} \frac{A_i x_i^{m-l}}{\kappa - x_i}, \qquad (3.13)$$

that is, we have

$$I_m = -\sum_{l=0}^{m-1}(-1)^l C_l^{m-1} \kappa^l \sum_{i,i\neq 0} A_i x_i^{m-l-1} - \sum_{l=1}^{m}(-1)^l C_l^m \kappa^l I_{m-l}. \qquad (3.14)$$

Therefore, I_m can be calculated recursively, if we are able to calculate the term $\sum_{i,i\neq 0} A_i x_i^{m-l-1}$ for $l = 0, 1, \dots, m-1$ appearing in (3.13). For example, for $m = 1$, we obtain

$$I_1 = \sum_{i,i\neq 0} \frac{A_i x_i}{\kappa - x_i} = -\sum_{i,i\neq 0} A_i + \kappa I_0 = -1 + A_0 + \kappa \sum_{i,i\neq 0} \frac{A_i}{\kappa - x_i}. \qquad (3.15)$$

In order to evaluate the sum $\sum_{i,i\neq 0} A_i x_i^k$, we rescale the roots x_i as ξx_i, where ξ is some nonzero constant. Then, we will have

$$\frac{1}{\xi} G_\mu(x/\xi) = \sum_i \frac{A_i}{x - \xi x_i} + \frac{A_0}{x - \xi x_0}. \qquad (3.16)$$

Now, we take the mth derivative of (3.16) to obtain

$$\frac{\partial^m}{\partial \xi^m}\left(\frac{1}{\xi} G_\mu(x/\xi)\right) = m!\left(\sum_{i,i\neq 0} \frac{A_i x_i^m}{(x - \xi x_i)^{m+1}} + \frac{A_0 x_0^m}{(x - \xi x_0)^{m+1}}\right), \qquad (3.17)$$

where at the limit of the large x, one can obtain the following simple form:

$$\lim_{x\to\infty} \frac{\partial^m}{\partial \xi^m}\left(\frac{1}{\xi} G_\mu(x/\xi)\right) = m!\left(\frac{\sum_{i,i\neq 0} A_i x_i^m + A_0 x_0^m}{x^{m+1}}\right). \qquad (3.18)$$

Therefore, we obtain

$$\sum_{i,i\neq 0} A_i x_i^m = \frac{1}{m!} \lim_{x\to\infty}\left[x^{m+1} \frac{\partial^m}{\partial \xi^m}\left(\frac{1}{\xi} G_\mu(x/\xi)\right)\right] - A_0 x_0^m. \qquad (3.19)$$

3.1. Two-point resistances up to the third stratum

In this subsection, we give analytical formulae for two-point resistances $R_{\alpha\beta^{(i)}}$, $i = 1, 2, 3$, in terms of the intersection numbers of the resistor networks.

It should be noted that, for two arbitrary nodes α and β such that $\beta \in R_1(\alpha)$, we have $P_1'(x) = \frac{x}{\sqrt{\kappa}}$. Therefore, by using (3.10) and (3.15), we obtain

$$L_{\beta\alpha}^{-1} = \frac{r}{\kappa} \sum_{i,i\neq 0} \frac{A_i x_i}{\kappa - x_i} = \frac{-r}{\kappa} \sum_{i,i\neq 0} A_i + r \sum_{i,i\neq 0} \frac{A_i}{\kappa - x_i}. \qquad (3.20)$$

Therefore, by using (3.8), we obtain the following simple result for all $\beta \in R_1(\alpha)$:

$$R_{\alpha\beta^{(1)}} = \frac{2r}{\kappa} \sum_{i,i\neq 0} A_i = \frac{2r}{\kappa}(1 - A_0) = \frac{2r}{\kappa}\left(1 - \frac{1}{v}\right), \qquad (3.21)$$

where v is the number of vertices of the graph, and in the last equality we have used the fact that for regular graphs, we have

$$A_0 = \frac{1}{v}. \tag{3.22}$$

In the following, we give analytical formulae for calculating two-point resistances $R_{\alpha\beta^{(2)}}$ and $R_{\alpha\beta^{(3)}}$, where $R_{\alpha\beta^{(2)}}(R_{\alpha\beta^{(3)}})$ denotes the mutual resistances between α and all $\beta \in R_2(\alpha)$ (all $\beta \in R_3(\alpha)$).

By using (2.38) and that $P'_k = \frac{Q_k}{\sqrt{\omega_1 \dots \omega_k}}$, we have $P'_2(x) = \frac{1}{\sqrt{\omega_1 \omega_2}}(x^2 - \alpha_1 x - \omega_1)$. Then, from (3.10) after some simplifications we obtain for $\beta \in R_2(\alpha)$:

$$L^{-1}_{\alpha\beta^{(2)}} = \frac{r}{\sqrt{\omega_1 \omega_2 \kappa_2}} \left(- \sum_{i,i \neq 0} A_i x_i + (\alpha_1 - \kappa) \sum_{i,i \neq 0} A_i + \kappa(\kappa - \alpha_1 - 1) \sum_{i,i \neq 0} \frac{A_i}{\kappa - x_i} \right). \tag{3.23}$$

By substituting $\alpha_1 = \kappa - b_1 - c_1$ in $\kappa(\kappa - \alpha_1 - 1)$, we obtain

$$\kappa(\kappa - \alpha_1 - 1) = \kappa(b_1 + c_1 - 1) = \kappa b_1. \tag{3.24}$$

Then, the coefficient of the term $\sum_{i,i \neq 0} \frac{A_i}{\kappa - x_i}$ in (3.23) is

$$\frac{r \kappa b_1}{\sqrt{\omega_1 \omega_2 \kappa_2}} = \frac{r \kappa b_1}{\sqrt{\kappa b_1 c_2 \kappa_2}} = r \sqrt{\frac{\kappa b_1}{c_2 \kappa_2}} = r. \tag{3.25}$$

Therefore, (3.23) can be written as

$$L^{-1}_{\alpha\beta^{(2)}} = \frac{r}{\sqrt{\omega_1 \omega_2 \kappa_2}} \left(- \sum_{i,i \neq 0} A_i x_i + (\alpha_1 - \kappa) \sum_{i,i \neq 0} A_i \right) + r \sum_{i,i \neq 0} \frac{A_i}{\kappa - x_i}, \tag{3.26}$$

where the sum $\sum_{i,i \neq 0} A_i x_i$ can be calculated by using (3.19). It can be easily shown that

$$\lim_{x \to \infty} \left[x^2 \frac{\partial}{\partial \xi} \left(\frac{1}{\xi} G_\mu(x/\xi) \right) \right] = a_{d-2} - b_{d-1}, \tag{3.27}$$

where a_{d-2} and b_{d-1} are the coefficients of x^{d-2} and x^{d-1} in $Q^{(1)}_{d-1}$ and Q_d, respectively. From the recursion relations (2.38) and (2.40), one can see that $a_{d-2} = b_{d-1} = -(\alpha_1 + \cdots + \alpha_d)$. Therefore, from (3.19) and (3.27), we obtain

$$\sum_{i,i \neq 0} A_i x_i = -A_0 \kappa = -\frac{\kappa}{v}. \tag{3.28}$$

Then, by using (3.7) and (3.26), one can write $R_{\alpha\beta^{(2)}}$ as follows:

$$R_{\alpha\beta^{(2)}} = \frac{2r}{\sqrt{\omega_1 \omega_2 \kappa_2}} \left\{ (\kappa - \alpha_1) - \frac{2\kappa - \alpha_1}{v} \right\}, \tag{3.29}$$

where by using (2.17) and (2.22), we obtain the following main result in terms of the intersection numbers of the graph:

$$R_{\alpha\beta^{(2)}} = \frac{2r}{b_0 b_1} \left\{ b_1 + 1 - \frac{b_0 + b_1 + 1}{v} \right\}. \tag{3.30}$$

Now, consider $\beta \in R_3(\alpha)$. Then, by using (2.38) and $P'_k = \frac{Q_k}{\beta_1, \dots, \beta_k}$, we obtain $P'_3(x) = \frac{1}{\sqrt{\omega_1 \omega_2 \omega_3}}(x^3 - (\alpha_1 + \alpha_2)x^2 - (\omega_1 + \omega_2 - \alpha_1 \alpha_2)x + \alpha_2 \omega_1)$. As above, after some

calculations, we obtain for $\beta \in R_3(\alpha)$:

$$L_{\alpha\beta^{(3)}}^{-1} = \frac{r}{\sqrt{\omega_1\omega_2\omega_3\kappa_3}}\left\{\frac{\kappa^2}{v} - \left(a_{d-3} - b_{d-2} + b_{d-1}^2 - b_{d-1}a_{d-2}\right) - (\alpha_1 + \alpha_2 - \kappa)\frac{\kappa}{v}\right.$$
$$- (\kappa^2 - \kappa(\alpha_1 + \alpha_2) - \omega_1 - \omega_2 + \alpha_1\alpha_2)\left(\frac{v-1}{v}\right)$$
$$\left. + (\kappa^3 - \kappa^2(\alpha_1 + \alpha_2) - \kappa(\omega_1 + \omega_2 - \alpha_1\alpha_2) + \alpha_2\omega_1)\sum_{i,i\neq 0}\frac{A_i}{\kappa - x_i}\right\}. \quad (3.31)$$

Again, by substituting $\alpha_1, \alpha_2, \omega_1$ and ω_2 from (2.22), we have

$$\frac{1}{\sqrt{\omega_1\omega_2\omega_3\kappa_3}}(\kappa^3 - \kappa^2(\alpha_1 + \alpha_2) - \kappa(\omega_1 + \omega_2 - \alpha_1\alpha_2) + \alpha_2\omega_1) = \frac{\kappa b_1 b_2}{\sqrt{\kappa b_1 c_2 b_2 c_3 \kappa_3}} = 1. \quad (3.32)$$

Therefore, (3.31) can be written as follows:

$$L_{\alpha\beta^{(3)}}^{-1} = \frac{r}{\sqrt{\omega_1\omega_2\omega_3\kappa_3}}\left\{\frac{\kappa^2}{v} - \left(a_{d-3} - b_{d-2} + b_{d-1}^2 - b_{d-1}a_{d-2}\right) - (\alpha_1 + \alpha_2 - \kappa)\frac{\kappa}{v}\right.$$
$$\left. - (\kappa^2 - \kappa(\alpha_1 + \alpha_2) - \omega_1 - \omega_2 + \alpha_1\alpha_2)\left(\frac{v-1}{v}\right)\right\} + r\sum_{i,i\neq 0}\frac{A_i}{\kappa - x_i}. \quad (3.33)$$

In (3.31), we have used the following equality:

$$\lim_{x^2\to\infty}\left[x^3\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{\xi}G_\mu(x/\xi)\right)\right] = 2\left(a_{d-3} - b_{d-2} + b_{d-1}^2 - b_{d-1}a_{d-2}\right), \quad (3.34)$$

where a_{d-3} and b_{d-2} are the coefficients of x^{d-3} and x^{d-2} in $Q_{d-1}^{(1)}$ and Q_d, respectively. From the recursion relations (2.38) and (2.40), one can see that $a_{d-3} = \prod_{i<j=1}^{d}\alpha_i\alpha_j - (\omega_1 + \cdots + \omega_{d-1})$ and $b_{d-2} = a_{d-3} - \omega_d$. Therefore, we have $a_{d-3} - b_{d-2} + b_{d-1}^2 - b_{d-1}a_{d-2} = \omega_d$.

Then, by using (3.7), $R_{\alpha\beta^{(3)}}$ is given by

$$R_{\alpha\beta^{(3)}} = \frac{2r}{\sqrt{\omega_1\omega_2\omega_3\kappa_3}}\left\{\omega_d + (\alpha_1 + \alpha_2 - 2\kappa)\frac{\kappa}{v} + (\kappa^2 - \kappa(\alpha_1 + \alpha_2))\right.$$
$$\left. - \omega_1 - \omega_2 + \alpha_1\alpha_2)\left(\frac{v-1}{v}\right)\right\}. \quad (3.35)$$

In terms of the intersection numbers of the graph, we obtain the following main result:

$$R_{\alpha\beta^{(3)}} = \frac{2r}{b_0 b_1 b_2}\left\{b_{d-1}c_d + b_2 - b_0 + c_2 + b_1 b_2 - \frac{(b_0 + 1)(b_2 + c_2) + b_1(b_0 + b_2)}{v}\right\}. \quad (3.36)$$

4. Examples

In this section, we calculate two-point resistances $R_{\alpha\beta^{(i)}}, i = 1, 2, 3$, by using (3.21), (3.30) and (3.36), in some important distance-regular networks with diameters $d = 1, d = 2$ and $d > 2$, respectively.

4.1. Complete network K_v

The complete network K_v is the simplest example of distance-regular networks. This network has v vertices with $v(v-1)/2$ edges; the degree of each vertex is $\kappa = v - 1$ also the network has diameter $d = 1$. The intersection array of the network is $\{b_0; c_1\} = \{v - 1; 1\}$. Clearly,

this graph has only two strata $R_0(\alpha) = \alpha$ and $R_1(\alpha) = \{\beta : \beta \neq \alpha\}$. Then, we have only one kind of two-point resistances which is obtained by using (3.21) as follows:

$$R_{\alpha\beta^{(1)}} = \frac{2r}{v-1}\left(1 - \frac{1}{v}\right) = \frac{2r}{v} \qquad \text{for all} \quad \beta \in R_1(\alpha). \tag{4.1}$$

4.2. Strongly regular networks

One of the most important distance-regular networks is that with the diameter $d = 2$, called the strongly regular network. A network with v vertices is strongly regular with parameters v, κ, λ, μ whenever it is not complete or edgeless and

 (i) each vertex is adjacent to κ vertices;
 (ii) for each pair of adjacent vertices there are λ vertices adjacent to both, and
 (iii) for each pair of non-adjacent vertices there are μ vertices adjacent to both.

For a strongly regular network, the intersection array is given by

$$\{b_0, b_1; c_1, c_2\} = \{\kappa, \kappa - \lambda - 1; 1, \mu\}. \tag{4.2}$$

One can note that if we consider networks with diameter 2 and maximum degree κ and $\alpha \in V$, then α has at most κ neighbours, and at most $\kappa(\kappa - 1)$ vertices lie at a distance 2 from α. Therefore,

$$v \leqslant 1 + \kappa + \kappa^2 - \kappa = \kappa^2 + 1, \quad \text{or} \quad \kappa \geqslant \sqrt{v - 1}, \tag{4.3}$$

where in the following by using inequality (4.3) we will obtain upper bounds for two-point resistances in strongly regular networks. To do so, first we calculate two-point resistances for these networks.

By using (3.21), (3.30) and (4.2), we obtain

$$R_{\alpha\beta^{(1)}} = \frac{2r}{\kappa}\left(\frac{v-1}{v}\right), \tag{4.4}$$

and

$$R_{\alpha\beta^{(2)}} = \frac{2r}{\kappa(\kappa - \lambda - 1)}\left(\kappa - \lambda - \frac{2\kappa - \lambda}{v}\right), \tag{4.5}$$

respectively. Then, from (4.3) and (4.4), we obtain the following upper bound for $R_{\alpha\beta^{(1)}}$:

$$R_{\alpha\beta^{(1)}} \leqslant \frac{2r\sqrt{v-1}}{v}. \tag{4.6}$$

Now, we consider the following two well-known strongly regular networks.

(A) *Petersen network*
A Petersen network [14] is a strongly regular network with parameters $(v, \kappa, \lambda, \eta) = (10, 3, 0, 1)$ and the intersection array $\{b_0, b_1; c_1, c_2\} = \{3, 2; 1, 1\}$. Therefore, by using (4.4) and (4.5), we obtain

$$R_{\alpha\beta^{(1)}} = \frac{3r}{5} \qquad \text{and} \qquad R_{\alpha\beta^{(2)}} = \frac{4r}{5}. \tag{4.7}$$

From (4.7), it is seen that $R_{\alpha\beta^{(1)}}$ in the Petersen graph saturates the upper bound (4.6).

(B) *Normal subgroup scheme*

Definition 2.3. *The partition $P = \{P_0, P_1, \ldots, P_d\}$ of a finite group G is called a blueprint [14] if*

(i) $P_0 = \{e\}$

(ii) for $i = 1, 2, \ldots, d$ if $g \in P_i$ then $g^{-1} \in P_i$

(iii) the set of relations $R_i = \{(\alpha, \beta) \in G \otimes G | \alpha^{-1}\beta \in P_i\}$ on G form an association scheme.

The set of real conjugacy classes P_i given in appendix A of [1] is an example of blueprint on G. Also, one can show that in the regular representation, the class sums $\bar{P}_i$ for $i = 0, 1, \ldots, d$ defined as

$$\bar{P}_i = \sum_{\gamma \in P_i} \gamma \in CG, \qquad i = 0, 1, \ldots, d \tag{4.8}$$

are the adjacency matrices of a blueprint scheme.

In [1], it has been shown that, if H be a normal subgroup of G, the following blueprint classes,

$$P_0 = \{e\}, \qquad P_1 = G - \{H\}, \qquad P_2 = H - \{e\}, \tag{4.9}$$

define a strongly regular network with parameters $(v, \kappa, \lambda, \eta) = (g, g - h, g - 2h, g - h)$ and the following intersection array:

$$\{b_0, b_1; c_1, c_2\} = \{g - h, h - 1; 1, g - h\}, \tag{4.10}$$

where $g := |G|$ and $h := |H|$. It is interesting to note that in a normal subgroup scheme, the intersection array and other parameters depend only on the cardinalities of the group and its normal subgroup. From (4.9), it is seen that $|R_2(\alpha)| = h - 1$. Then, by using (4.4) and (4.5), we obtain

$$R_{\alpha\beta^{(1)}} = \frac{2r(g-1)}{g(g-h)}, \qquad \text{and} \qquad R_{\alpha\beta^{(2)}} = \frac{2r}{(g-h)}. \tag{4.11}$$

One should note that, the maximum degree κ for the normal subgroup scheme is $\kappa_{\max} = g - 2$ ($h = 2$), which can be appear in networks with even cardinality such as the dihedral group. Therefore, for the normal subgroup scheme (strongly regular networks with parameters $(g, g - h, g - 2h, g - h)$), we have

$$\kappa \leqslant g - 2, \tag{4.12}$$

and therefore, by using (4.3), (4.11) and (4.12) ($\kappa = g - h$), we obtain upper and lower bounds for $R_{\alpha\beta^{(1)}}$ and $R_{\alpha\beta^{(2)}}$ as follows:

$$\frac{2r(g-1)}{g(g-2)} \leqslant R_{\alpha\beta^{(1)}} \leqslant \frac{2r\sqrt{g-1}}{g}, \qquad \frac{2r}{g-2} \leqslant R_{\alpha\beta^{(2)}} \leqslant \frac{2r}{\sqrt{g-1}}. \tag{4.13}$$

As an example, we consider the dihedral group $G = D_{2m}$, where its normal subgroup is $H = Z_m$. Therefore, the blueprint classes are given by

$$P_0 = \{e\}, \qquad P_1 = \{b, ab, a^2b, \ldots, a^{m-1}b\}, \qquad P_2 = \{a, a^2, \ldots, a^{(m-1)}\}, \tag{4.14}$$

which form a strongly regular network with parameters $(2m, m, 0, m)$ and the following intersection array:

$$\{b_0, b_1; c_1, c_2\} = \{m, m - 1; 1, m\}. \tag{4.15}$$

By using (4.11), we obtain

$$R_{\alpha\beta^{(1)}} = \frac{r(2m-1)}{m^2}, \qquad \text{and} \qquad R_{\alpha\beta^{(2)}} = \frac{2r}{m}. \tag{4.16}$$

4.3. Cycle network C_v

As was defined in subsection 2.1, the cycle network or cycle with v vertices is denoted by C_v with $\kappa = 2$. We consider $v = 2m$ (the case $v = 2m + 1$ can be considered similarly). Then, by substituting the intersection array given by (2.26) in (3.21), (3.30) and (3.36) respectively, we obtain

$$R_{\alpha\beta^{(1)}} = r\left(\frac{2m-1}{2m}\right), \qquad R_{\alpha\beta^{(2)}} = 2r\left(\frac{m-1}{m}\right) \qquad \text{and} \qquad R_{\alpha\beta^{(3)}} = 3r\left(\frac{2m-3}{2m}\right).$$
(4.17)

From (4.17), one can easily deduce that

$$R_{\alpha\beta^{(k)}} = kr\left(\frac{2m-k}{2m}\right) \qquad k = 1, 2, \ldots, m.$$
(4.18)

4.4. d-cube

The d-cube, i.e. the hypercube of dimension d, also called the Hamming cube, is a network with 2^d nodes, each of which can be labelled by a d-bit binary string. Two nodes on the hypercube described by bitstrings $\vec{x}$ and $\vec{y}$ are connected by an edge if $|\vec{x} - \vec{y}| = 1$, where $|\vec{x}|$ is the Hamming weight of $\vec{x}$. In other words, if $\vec{x}$ and $\vec{y}$ differ by only a single-bit flip, then the two corresponding nodes on the graph are connected. Thus, each of the 2^d nodes on the d-cube has degree d. For the d-cube, we have $d + 1$ strata with

$$\kappa_i = \frac{d!}{i!(d-i)!}, \qquad 0 \leqslant i \leqslant d-1.$$
(4.19)

The intersection numbers are given by

$$b_i = d - i, \qquad 0 \leqslant i \leqslant d-1; \qquad c_i = i, \qquad 1 \leqslant i \leqslant d.$$
(4.20)

Then, by using (3.21), (3.30) and (3.36), we obtain

$$R_{\alpha\beta^{(1)}} = \frac{2^d - 1}{d2^{d-1}}r, \qquad R_{\alpha\beta^{(2)}} = \frac{2^{d-1} - 1}{(d-1)2^{d-2}}r, \qquad \text{and}$$

$$R_{\alpha\beta^{(3)}} = \frac{r}{d(d-1)(d-2)}\left\{\frac{2^d(d^2 - 2d + 2) - 3d(d-1) - 2}{2^{d-1}}\right\}.$$
(4.21)

4.5. Johnson network

Let $n \geqslant 2$ and $d \leqslant n/2$. The Johnson network $J(n, d)$ has all d-element subsets of $\{1, 2, \ldots, n\}$ such that two d-element subsets are adjacent if their intersection has size $d - 1$. Two d-element subsets are then at distance i if and only if they have exactly $d - i$ elements in common. The Johnson network $J(n, d)$ has $v = \frac{n!}{d!(n-d)!}$ vertices, diameter d and the valency $\kappa = d(n - d)$. Its intersection array is given by

$$b_i = (d - i)(n - d - i), \qquad 0 \leqslant i \leqslant d-1; \qquad c_i = i^2, \qquad 1 \leqslant i \leqslant d.$$
(4.22)

Then, by using (3.21), (3.30) and (3.36), we obtain

$$R_{\alpha\beta^{(1)}} = \frac{2(n! - d!(n-d)!)}{d(n-d)n!}r,$$

$$R_{\alpha\beta^{(2)}} = \frac{2r}{d(d-1)(n-d)(n-d-1)}$$

$$\times \left\{d(n-d) - (n-2) + \frac{d!(n-d)!(n-2-2d(n-d))}{n!}\right\}$$

and

$$R_{\alpha\beta^{(3)}} = \frac{2r}{d(d-1)(d-2)(n-d)(n-d-1)(n-d-2)}$$
$$\times \left\{ d^2(n-2d+1) + (3n-2d(n-d)-10)\frac{d(n-d)d!(n-d)!}{n!} \right.$$
$$+ [d^2(n-d)^2 - d(n-d)(3n-9) - 4(d-1)(n-d-1)$$
$$\left. + 2(n-2)(n-4)]\left(1 - \frac{d!(n-d)!}{n!}\right) \right\}. \tag{4.23}$$

5. Two-point resistances in infinite regular networks

As results (3.8) and (3.19) show, the two-point resistances on a network depend only on the Stieltjes function $G_\mu(x)$ corresponding to the network. Clearly, the Stieltjes function corresponding to an infinite network possesses a unique representation as an infinite continued fraction as follows:

$$G_\mu(x) = \int_R \frac{\mu(dy)}{x-y} = \cfrac{1}{x-\alpha_0 - \cfrac{\beta_1^2}{x-\alpha_1 - \cfrac{\beta_2^2}{x-\alpha_2 - \cfrac{\beta_3^2}{x-\alpha_3-\cdots}}}}, \tag{5.1}$$

where the sequence $\alpha_0, \alpha_1, \ldots; \beta_1, \beta_2, \ldots$ never stops. One should note that in the cases for which the QD parameters α_i and β_i iterate themselves after some finite steps, one can find a closed form for the infinite continued fraction (5.1). This situation takes place, for instance, in the infinite line network which we will consider as an example in the following. But in most cases, this situation dose not occur and one cannot obtain a closed form for the Stieltjes function of the network. In these cases, we approximate the infinite networks with the best finite ones (it is well known from the theory of continued fractions that all infinite (convergent) continued fraction expansions can be approximated with some finite ones; for more details see appendix B). That is, in the infinite networks for which the Stieltjes function $G_\mu(x)$ dose not possess a closed form, one can evaluate the best finite number n such that $G_\mu(x)$ converges to $\frac{Q_{n-1}^{(1)}(x)}{Q_n(x)}$. To this end, we compare two-point resistances in the finite networks with those of the corresponding infinite ones. More clearly, we plot the curve of one of the two-point resistances such as $R_{\alpha\beta^{(1)}}$ on the corresponding finite network in terms of the size v of the network and estimate the best approximation for v such that the difference $|R_{\alpha\beta^{(1)}} - \lim_{v\to\infty} R_{\alpha\beta^{(1)}}|$ tends to zero. One can use this fact in order to approximate the two-point resistances on an infinite-size resistor network with those of the corresponding finite-size network.

In the following, we give some examples of infinite networks for which two-point resistances are evaluated exactly or approximately.

5.1. Examples

(1) *Infinite line network*
Obviously, the infinite line network is the large-size limit ($v \to \infty$) of the cycle network C_v discussed before. Therefore, the QD parameters of the infinite line network are given by $\alpha_1 = \alpha_2 = \cdots = 0; \omega_1 = 2, \omega_1 = \omega_2 = \cdots = 1$. Then, by using (5.1), the Stieltjes function for the infinite line network reads as

$$G_\mu(x) = \cfrac{1}{x - \cfrac{2}{x - \cfrac{1}{x - \cfrac{1}{x-\cdots}}}} = \frac{1}{x - 2G'(x)}, \tag{5.2}$$

where

$$G'(x) = \frac{1}{x - \frac{1}{x - \frac{1}{x - \cdots}}} = \frac{1}{x - G'(x)}. \tag{5.3}$$

From (5.3), $G'(x)$ is easily obtained as

$$G'(x) = \frac{x \pm \sqrt{x^2 - 4}}{2}. \tag{5.4}$$

By substituting (5.4) in (5.2), we obtain the Stieltjes function as follows:

$$G_\mu(x) = \frac{1}{\sqrt{x^2 - 4}}. \tag{5.5}$$

Then, for A_0 we have

$$A_0 = \lim_{x \to 2}((x - 2)G_\mu(x)) = 0. \tag{5.6}$$

Therefore, by using (3.21), (3.30) and (3.36), we obtain

$$R_{\alpha\beta^{(1)}} = r, \qquad R_{\alpha\beta^{(2)}} = 2r, \qquad R_{\alpha\beta^{(3)}} = 3r. \tag{5.7}$$

In fact, it can be easily shown that

$$R_{\alpha\beta^{(k)}} = kr, \qquad k = 1, 2, \ldots, \tag{5.8}$$

where this result could be obtained from (4.18), for large m.

(2) *d-cube in the limit $d \to \infty$*

By using (2.22) and (4.20), one can obtain QD parameters of the d-cube as follows:

$$\alpha_i = 0, \qquad \omega_i = i(d - i + 1); \qquad i = 0, 1, 2, \ldots, \tag{5.9}$$

where we have used $\kappa = \kappa_1 = d$. Then by using (2.39), the Stieltjes function reads as

$$G_\mu(x) = \frac{1}{x - \frac{d}{x - \frac{2(d-1)}{x - \frac{3(d-2)}{x - \cdots}}}}, \tag{5.10}$$

which is not a closed form. From (4.21), one can see that at the limit of the large dimension d, the two-point resistances $R_{\alpha\beta^{(i)}}, i = 1, 2, 3$, tend to zero. Therefore, we find the best approximation for d such that $R_{\alpha\beta^{(i)}} \sim 0$. By plotting one of the $R_{\alpha\beta^{(i)}}$, say $R_{\alpha\beta^{(1)}}$ given by (4.21), in terms of d one can see that $R_{\alpha\beta^{(1)}}$ tend to zero for d larger than ~ 200. Therefore, the finite d cube with d larger than ~ 200 is a good approximation for the infinite hypercube resistor network.

(3) *Johnson network in the limit $n \to \infty$*

By using (2.22) and (4.20), one can obtain QD parameters of the Johnson network $J(n, d)$ as follows:

$$\alpha_i = i(n - 2i), \qquad \omega_i = i^2(d - i + 1)(n - d - i + 1); \qquad i = 0, 1, 2, \ldots. \tag{5.11}$$

Then by using (2.39), the Stieltjes function reads as

$$G_\mu(x) = \frac{1}{x - \frac{d(n-d)}{x - (n-2) - \frac{4(d-1)(n-d-1)}{x - 2(n-4) - \frac{9(d-2)(n-d-2)}{x - \cdots}}}}. \tag{5.12}$$

Clearly, $G_\mu(x)$ is not a closed form for a given d, where $n \to \infty$ and so we should approximate it with a suitable finite one. To do so, one should note that for a given d, result (4.23) shows that, at the limit of the large dimension n, the two-point resistances $R_{\alpha\beta^{(i)}}, i = 1, 2, 3$, tend to zero. Since $R_{\alpha\beta^{(1)}}$ tend to zero for n larger than ~ 120. Therefore, the finite Johnson network $J(n, d)$ with n larger than ~ 120 is a good approximation for the infinite Johnson resistor network.

6. Two-point resistances in more general networks

Although we discussed through the paper only the case of distance-regular networks, the method can also be used for any arbitrary regular network. For calculating two-point resistances, we need only to know the Stieltjes function $G_\mu(x)$. For two arbitrary nodes α and β of the network, we choose one of the nodes, say α, as a reference vertex. Then, the Stieltjes function $G_\mu(x)$ can be calculated by using the recursion relations (2.38) and (2.40), where, as has been shown in [26], the coefficients α_i and β_i, for $i = 1, \ldots, d$ in the recursion relations are obtained by applying the Lanczos algorithm to the adjacency matrix of the network and the reference vertex $|\alpha\rangle$. In fact, the adjacency matrix of the network takes a tridiagonal form in the orthonormal basis $\{|\phi_i\rangle, i = 0, 1, \ldots, d\}$ produced by the Lanczos algorithm, and so we obtain again three-term recursion relations as (2.38). But, in general, the basis produced by the Lanczos algorithm does not define a stratification basis, in the sense that, a vertex ket $|\beta\rangle$ of the network may appear in more than one of the base vectors $|\phi_i\rangle$. In these cases, if d is equal to v (the number of vertices of the network), one can write each vertex ket $|\beta\rangle$ uniquely as a superposition of the base vectors $|\phi_i\rangle$ and calculate two-point resistance $R_{\alpha\beta}$ by calculating the entries $\langle\phi_i|L^{-1}|\alpha\rangle$ for all $i = 0, 1, \ldots, d$ as illustrated through the paper. In the most cases, d is less than v. In these cases, we need to obtain some additional orthonormal base vectors $\{|\psi_i\rangle, i = 1, \ldots, v-d-1\}$ such that the new bases are orthogonal to the subspace spanned by $\{|\phi_i\rangle, i = 0, 1, \ldots, d\}$. One can obtain some such additional base vectors, by choosing a normalized vector orthogonal to the subspace spanned by $\{|\phi_i\rangle, i = 0, 1, \ldots, d\}$ as a new reference state and applying the Lanczos algorithm to the adjacency matrix of the network and the new reference state. If the number of the new orthonormal base vectors still be less than $v - d - 1$, we choose another normalized reference state orthogonal to the subspace spanned by all previous orthonormal bases and apply the Lanczos algorithm to the adjacency matrix and the new chosen reference state. By repeating this process until v orthonormal bases are obtained, one can solve a system of v equations with v unknowns to write each vertex ket $|\beta\rangle$ as a superposition of the v orthonormal bases.

7. Conclusion

The resistance between two arbitrary nodes in a distance-regular resistor network was obtained in terms of the Stieltjes transform of the spectral measure or Stieltjes function associated with the network and its derivatives. It was shown that the resistances between a node α and all nodes β belonging to the same stratum with respect to the α are the same. Also, explicit analytical formulae for two-point resistances $R_{\alpha\beta}$ for β belonging to the first, second and third stratum with respect to the α were driven in terms of the size of the network and the corresponding intersection numbers. In particular, the two-point resistances in a strongly regular network with parameters $(v, \kappa, \lambda, \mu)$ were given in terms of these parameters. Moreover, the lower and upper bounds for two-point resistances in strongly regular networks were discussed. It was discussed that the introduced method can be used not only for distance-regular networks, but also for any arbitrary regular network by employing the Lanczos algorithm iteratively.

Appendix A

In this appendix, we give the two-point resistances $R_{\alpha\beta^{(i)}}, i = 1, 2, 3$, for some important distance-regular networks with $v \leqslant 70$.

The network	v	Intersection array	Reference	$R_{\alpha\beta^{(1)}}$	$R_{\alpha\beta^{(2)}}$	$R_{\alpha\beta^{(3)}}$
Icosahedron	12	{5, 2, 1; 1, 2, 5}	[32]	$\frac{11r}{30}$	$\frac{7r}{15}$	$\frac{r}{2}$
L(Petersen)	15	{4, 2, 1; 1, 1, 4}	[32]	$\frac{7r}{15}$	$\frac{19r}{30}$	$\frac{2r}{3}$
Pappus, Three-cover $K_{3,3}$	18	{3, 2, 2, 1; 1, 1, 2, 3}	[32]	$\frac{17r}{27}$	$\frac{8r}{9}$	$\frac{26r}{27}$
Desargues	20	{3, 2, 2, 1, 1; 1, 1, 2, 2, 3}	[32]	$\frac{19r}{30}$	$\frac{8r}{9}$	$\frac{59r}{60}$
Dodecahedron	20	{3, 2, 1, 1, 1; 1, 1, 1, 2, 3}	[32]	$\frac{19r}{30}$	$\frac{9r}{10}$	$\frac{16r}{15}$
$GH(2,1)$	21	{4, 2, 2; 1, 1, 2}	[32]	$\frac{10r}{21}$	$\frac{2r}{3}$	$\frac{5r}{7}$
Klein	24	{7, 4, 1; 1, 2, 7}	[32]	$\frac{23r}{84}$	$\frac{9r}{28}$	$\frac{r}{3}$
$GQ(2,4)\backslash$spread	27	{8, 6, 1; 1, 3, 8}	[32]	$\frac{13r}{54}$	$\frac{29r}{108}$	$\frac{5r}{18}$
$H(3,3)$	27	{6, 4, 2; 1, 2, 3}	[32]	$\frac{26r}{81}$	$\frac{31r}{81}$	$\frac{11r}{27}$
Coxeter	28	{3, 2, 2, 1; 1, 1, 1, 2}	[32]	$\frac{9r}{14}$	$\frac{13r}{14}$	$\frac{73r}{84}$
Taylor($P(13)$)	28	{13, 6, 1; 1, 6, 13}	[33]	$\frac{27r}{182}$	$\frac{44r}{273}$	$\frac{91r}{546}$
Tutte's eight-cage	30	{3, 2, 2, 2; 1, 1, 1, 3}	[32]	$\frac{29r}{45}$	$\frac{14r}{15}$	$\frac{139r}{90}$
Taylor(GQ(2, 2))	32	{15, 8, 1; 1, 8, 15}	[33]	$\frac{31r}{240}$	$\frac{11r}{80}$	$\frac{17r}{120}$
Taylor($T(6)$)	32	{15, 6, 1; 1, 6, 15}	[33]	$\frac{31r}{240}$	$\frac{101r}{720}$	$\frac{13r}{90}$
$IG(AG(2,4)\backslash pc)$	32	{4, 3, 3, 1; 1, 1, 3, 4}	[34]	$\frac{31r}{64}$	$\frac{5r}{8}$	$\frac{125r}{192}$
Wells	32	{5, 4, 1, 1; 1, 1, 4, 5}	[34]	$\frac{31r}{80}$	$\frac{15r}{32}$	$\frac{39r}{80}$
Hadamard graph	32	{8, 7, 4, 1; 1, 4, 7, 8}	[35]	$\frac{31r}{128}$	$\frac{15r}{56}$	$\frac{249r}{896}$
Odd(4)	35	{4, 2, 1; 1, 1, 4}	[36, 37]	$\frac{17r}{35}$	$\frac{22r}{35}$	$\frac{242r}{315}$
Sylvester	36	{5, 4, 4; 1, 1, 4}	[36]	$\frac{7r}{18}$	$\frac{17r}{36}$	$\frac{181r}{240}$
Taylor($P(17)$)	36	{17, 8, 1; 1, 8, 17}	[33]	$\frac{35r}{306}$	$\frac{149r}{1224}$	$\frac{279r}{272}$
Three-cover $K_{6,6}$	36	{6, 5, 4, 1; 1, 2, 5, 6}	[33]	$\frac{35r}{108}$	$\frac{17r}{45}$	$\frac{211r}{540}$
$SRG\backslash$spread	40	{9, 6, 1; 1, 2, 9}	[38]	$\frac{13r}{60}$	$\frac{11r}{45}$	$\frac{r}{4}$
$Ho-Si_2(x)$	42	{16, 5, 1; 1, 1, 6}	[36]	$\frac{41r}{126}$	$\frac{8r}{21}$	$\frac{34r}{105}$
Mathon (Cycl(13, 3))	42	{13, 8, 1; 1, 4, 13}	[39]	$\frac{41r}{273}$	$\frac{89r}{546}$	$\frac{91r}{13}$
$GO(2,1)$	45	{4, 2, 2, 2; 1, 1, 1, 2}	[34]	$\frac{22r}{45}$	$\frac{32r}{45}$	$\frac{4r}{5}$
Three-cover $GQ(2,2)$	45	{6, 4, 2, 1; 1, 1, 4, 6}	[34]	$\frac{44r}{135}$	$\frac{107r}{270}$	$\frac{221r}{540}$
Hadamard graph	48	{12, 11, 6, 1; 1, 6, 11, 12}	[33]	$\frac{47r}{288}$	$\frac{23r}{132}$	$\frac{565r}{3168}$
$IG(AG(2,5)\backslash pc)$	50	{5, 4, 4, 1; 1, 1, 4, 5}	[34]	$\frac{49r}{125}$	$\frac{12r}{25}$	$\frac{123r}{250}$
Mathon (Cycl(16, 3))	51	{16, 10, 1; 1, 5, 16}	[39]	$\frac{25r}{204}$	$\frac{89r}{680}$	$\frac{68r}{510}$
$GH(3,1)$	52	{6, 3, 3; 1, 1, 2}	[40]	$\frac{51r}{156}$	$\frac{11r}{26}$	$\frac{23r}{52}$
Taylor(SRG(25, 12))	52	{25, 12, 1; 1, 12, 25}	[33]	$\frac{51r}{650}$	$\frac{319r}{3900}$	$\frac{13r}{156}$
Three-cover $K_{9,9}$	54	{9, 8, 6, 1; 1, 3, 8, 9}	[33]	$\frac{53r}{243}$	$\frac{13r}{54}$	$\frac{239r}{972}$
Gosset,Tayl(Schläfli)	56	{27, 10, 1; 1, 10, 27}	[33]	$\frac{55r}{756}$	$\frac{289r}{3780}$	$\frac{49r}{630}$
Taylor(Co-Schläfli)	56	{27, 16, 1; 1, 16, 27}	[33]	$\frac{55r}{756}$	$\frac{227r}{3024}$	$\frac{11r}{144}$
Perkel	57	{6, 5, 2; 1, 1, 3}	[36, 41]	$\frac{56r}{171}$	$\frac{22r}{57}$	$\frac{68r}{171}$
Mathon(Cycl(11, 5))	60	{11, 8, 1; 1, 2, 11}	[39]	$\frac{59r}{330}$	$\frac{32r}{165}$	$\frac{r}{5}$
Mathon(Cycl(19, 3))	60	{19, 12, 1; 1, 6, 19}	[39]	$\frac{59r}{570}$	$\frac{187r}{1710}$	$\frac{3151r}{570}$
Taylor($SRG(29, 14)$)	60	{29, 14, 1; 1, 14, 29}	[33]	$\frac{59r}{870}$	$\frac{214r}{3045}$	$\frac{29r}{406}$
$GH(2,2)$	63	{6, 4, 4; 1, 1, 3}	[40]	$\frac{62r}{189}$	$\frac{76r}{189}$	$\frac{271r}{504}$

The network	v	Intersection array	Reference	$R_{\alpha\beta^{(1)}}$	$R_{\alpha\beta^{(2)}}$	$R_{\alpha\beta^{(3)}}$
$H(3, 4)$, Doob	64	{9, 6, 3; 1, 2, 3}	[33]	$\frac{7r}{32}$	$\frac{r}{4}$	$\frac{25r}{96}$
Locally Petersen	65	{10, 6, 4; 1, 2, 5}		$\frac{64r}{325}$	$\frac{73r}{325}$	$\frac{49r}{156}$
Doro	68	{12, 10, 3; 1, 3, 8}		$\frac{67r}{408}$	$\frac{145r}{816}$	$\frac{253r}{1020}$
Doubled odd(4)	70	{4, 3, 3, 2, 2, 1, 1; 1, 1, 2, 2, 3, 3, 4}	[33]	$\frac{69r}{140}$	$\frac{68r}{105}$	$\frac{869r}{1260}$
$J(8, 4)$	70	{16, 9, 4, 1; 1, 4, 9, 16}	[33]	$\frac{69r}{560}$	$\frac{337r}{2520}$	$\frac{691r}{5040}$

Appendix B

In this appendix, we recall some facts about the approximation of an infinite continued fraction as in (5.1) with a finite one. To do so, we use the following notation:

$$\cfrac{1}{(x - \alpha_0)-} \cfrac{\beta_1^2}{(x - \alpha_1)-} \cfrac{\beta_2^2}{(x - \alpha_2)-} \cfrac{\beta_3^2}{(x - \alpha_3)-} \cdots$$

for the infinite continued fraction in (5.1). Then, we consider the convergents C_n as

$$C_n = \cfrac{1}{(x - \alpha_0)-} \cfrac{\beta_1^2}{(x - \alpha_1)-} \cdots \cfrac{\beta_{n-1}^2}{(x - \alpha_{n-1})} = \frac{Q_{n-1}^{(1)}(x)}{Q_n(x)} \tag{B.1}$$

for $n \geqslant 1$, where $Q_n(x)$ and $Q_{n-1}^{(1)}(x)$ are polynomials with recursion relations (2.38) and (2.40), respectively. We say that an infinite continued fraction $\frac{1}{(x-\alpha_0)-}$ $\frac{\beta_1^2}{(x-\alpha_1)-} \frac{\beta_2^2}{(x-\alpha_2)-} \frac{\beta_3^2}{(x-\alpha_3)-} \cdots$ converges if $(C_n,$ is determinate and finite for all sufficiently large n and) $\lim_{n\to\infty} C_n$ exists. Of course, the value of the continued fraction is defined to be this limit, i.e.,

$$\cfrac{1}{(x - \alpha_0)-} \cfrac{\beta_1^2}{(x - \alpha_1)-} \cfrac{\beta_2^2}{(x - \alpha_2)-} \cfrac{\beta_3^2}{(x - \alpha_3)-} \cdots \equiv \lim_{n\to\infty} C_n. \tag{B.2}$$

One should note that similar to the convergent infinite series, the most infinite continued fractions (even if they converge) have not a closed form. But one can approximate such infinite continued fractions with some suitable finite ones. For example, every best rational approximation of $G_\mu(x)$ is a convergent $\frac{Q_{n-1}^{(1)}(x)}{Q_n(x)}$ for some n (see theorem 15, p 22 and theorem 16, p 24 of [13]).

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